

THE ISOMORPHISM PROBLEM FOR HIGMAN-THOMPSON GROUPS

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ABSTRACT. We prove that the Higman-Thompson groups $G_{n,r}^+$ and $G_{m,s}^+$ are isomorphic if and only if $m = n$ and $\gcd(n-1, r) = \gcd(n-1, s)$.

INTRODUCTION

Finitely presented groups are one of the most important classes of infinite groups, both by its ubiquity (e.g. they are fundamental groups of compact manifolds) and by the number of interesting subclasses it contains (hyperbolic groups and automatic groups, among others). Also they have interest in connection with algorithmic properties, as showed by Higman [4], who stated that a finitely generated group is embeddable in a finitely presented group if and only if is recursively presented. As simple groups are one of the milestones in the development of Group Theory, a fundamental topic in the study of groups is that of finitely presented simple groups.

The study of finitely presented simple groups began with Thompson's discovering in 1965 of the firsts two infinite examples in this class [12], now known as $G_{2,1}$ and $T_{2,1}$. In 1974 Higman [5] constructed a countably infinite family of finitely presented simple groups generalizing Thompson's group $G_{2,1}$. These are the commutator subgroups $G_{n,r}^+$ of the groups $G_{n,r}$ introduced in the same paper. There are various ways of describing these groups: automorphism groups of r -generated free algebras in the variety of algebras of sets that are in bijection with its own n -th direct power [5], groups of piecewise linear homeomorphisms of the unit interval with prescribed slopes and limited sets of non-differentiable points [11], groups of tree diagrams of finite n -ary r -forests [3], or groups of maximal inescapable (cofinite) isomorphisms [10].

Unfortunately, these groups are still not fully understood. For example, the isomorphism problem is not completely solved in this class, even if Higman [5] highlighted a great part of it. Higman showed that showed that this family contains infinitely many isomorphism types. Also he showed that $G_{n,r}^+ \cong G_{m,s}^+$ implies that $m = n$ and $\gcd(r, n-1) = \gcd(s, n-1)$ [5, Theorem 6.4], while the converse is known only for some particular cases (e.g. when $r \equiv s \pmod{n-1}$ [5, Section 3], or when $s = rc$ with c a divisor of n [5, Theorem 7.3]).

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In this paper we prove that the converse of [5, Theorem 6.4] holds: $G_{n,r}^+$ and $G_{m,s}^+$ are isomorphic if and only if $m = n$ and $\gcd(n-1, r) = \gcd(n-1, s)$. Hence, we close the isomorphism problem for this class. The key point for proving this result relies in the connection of this problem with a longstanding problem about isomorphisms of finitely presented algebras stated by Leavitt [6, 7], and recently solved by Abrams, Ánh and the author [1].

Now we summarize the contents of this paper. In Section 1 we recall the definition of $G_{n,r}$ following the lines of [10], and we list some properties enjoyed by these groups. In Section 2 we recall the definition of Leavitt algebras, and we quote [1, Theorem 4.14]:

Let d, n be positive integers, and K any field. Let $L_{K,n} = L_n$ denote the Leavitt algebra of type $(1, n-1)$ with coefficients in K . Then $L_n \cong M_d(L_n)$ if and only if $\gcd(d, n-1) = 1$.

Since the isomorphism is explicitly given in terms of the generators of the algebra, we use it in Section 3, where we relate $G_{n,r}$ with a group of invertible matrices in $M_r(L_n)$, and as a byproduct we give a proof of the converse of [5, Theorem 6.4].

1. BASICS ON HIGMAN-THOMPSON GROUPS

We will fix the essential definitions and results about Higman-Thompson groups that we will need in the sequel. Our sources are [3, 5, 9, 10].

Let $n, r \in \mathbb{N}$, $n \geq 2$ and $r \geq 1$, let $\mathcal{A}_n = \{a_1, \dots, a_n\}$ be an alphabet (through the rest of the paper we will assume $\mathcal{A}_n = \{1, \dots, n\}$ by defect), let $X_r = \{x_1, \dots, x_r\}$ a set of r elements disjoint of \mathcal{A}_n , and let W_n be the free monoid generated by \mathcal{A}_n and the empty word. Denote by $X_r W_n$ the set of finite words of the form $x_i \alpha$, where $\alpha \in W_n$.

Given $u, v \in X_r W_n$, we will denote $u \leq v$ if there exists $\alpha \in W_n$ such that $v = u\alpha$; notice that \leq is a partial order, and so $u < v$ means $u \leq v$ and $u \neq v$ with no ambiguity. We will say that a subset B of $X_r W_n$ is independent if its elements are pairwise \leq -incomparable.

A nonempty subset V of $X_r W_n$ is said to be a subspace if it is closed under right multiplication by elements of W_n . A subset B of a subspace V is a basis if it is independent and $V = B W_n$; a subset B of $X_r W_n$ is a basis if there exists a subspace V for which B is a basis. Notice that the set $B_V = \{y \in V \mid \text{no proper initial segment of } y \text{ belongs to } V\}$ is a basis for V , so that every subspace has a basis. In particular, X_r is a basis for $X_r W_n$.

A subspace V is cofinite if $|X_r W_n \setminus V| < \infty$. A basis B is cofinite if $V = B W_n$ is a cofinite subspace. Notice that then V is cofinite if it has a maximal finite basis. In particular, any finite basis is contained in a cofinite basis. Also it is clear that any finite intersection of cofinite subspaces is a cofinite subspace (this is [5, Corollary 1 to Lemma 2.4] stated in a different language).

If u is an element of $X_r W_n$, we will say that $\{ua_1, \dots, ua_n\}$ is a simple expansion of u . Given a basis B and $u \in B$ any element, $B' = (B \setminus \{u\}) \cup \{ua_1, \dots, ua_n\}$ is again a basis, that we call a simple expansion of B . Given B, C basis, we say that C is an expansion of B if there is a finite chain B_0, \dots, B_k of basis such that $B_0 = B$, $B_k = C$ and B_{i+1} is a simple expansion of B_i for every $0 \leq i \leq k-1$. Of course, any expansion of a cofinite basis is a cofinite basis as well.

An homomorphism θ between subspaces of $X_r W_n$ is a map satisfying $(uw)\theta = (u\theta)w$ for all $w \in W_n$, whenever $u\theta$ is defined. An isomorphism is a bijective homomorphism, and if the domain and the range of an isomorphism is cofinite, we say that it is a cofinite isomorphism.

An extension of a cofinite isomorphism θ is a cofinite isomorphism θ' such that $u\theta' = u\theta$, whenever $u\theta$ is defined. A cofinite isomorphism is maximal if it has no nontrivial extensions.

Now, we quote two fundamental facts:

Lemma 1.1. (c.f. [10, Lemma 1]) *Every cofinal isomorphism θ has a unique maximal extension θ^* .*

Let $\phi_i : U_i \rightarrow V_i$ be cofinite isomorphisms for $i = \{1, 2\}$. Now fix $S = V_1 \cap U_2$, $R = S\phi_1^{-1}$, $T = S\phi_2$, which are cofinite subspaces, and notice that $(\phi_1|_R) \circ (\phi_2|_S)$ is a cofinal isomorphism from R to S . So, we define $\phi_1\phi_2 := ((\phi_1|_R) \circ (\phi_2|_S))^*$. With this definition we have

Lemma 1.2. (c.f. [10, Lemma 2]) *The set of maximal cofinite isomorphisms is a group under the above defined operation.*

The group defined in Lemma 1.2 is the Higman-Thompson group $G_{n,r}$ defined originally in [5]. We introduce a the representation of the elements of $G_{n,r}$ which turns out to be a useful instrument to deal with the group.

Whenever $B = \{y_1, \dots, y_N\}$ and $C = \{z_1, \dots, z_N\}$ are expansions of X_r (and thus cofinite basis), the bijection

$$\begin{array}{ccc} \theta & B & \rightarrow C \\ & y_i & \mapsto z_i \end{array}$$

extends naturally to a cofinite isomorphism $\theta : BW_n \rightarrow CW_n$, so that $\theta \in G_{n,r}$. Thus, we can represent θ by the symbol

$$\theta = \begin{pmatrix} y_1 & \dots & y_N \\ z_1 & \dots & z_N \end{pmatrix}.$$

Conversely, every element $\theta \in G_{n,r}$ admits such a representation [5, Lemma 4.1].

Whenever

$$\varphi = \begin{pmatrix} x_1 & \dots & x_M \\ t_1 & \dots & t_M \end{pmatrix}$$

is a symbol for any other element in $G_{n,r}$, [5, Corollary 1 to Lemma 2.4] guarantees that there exists a common expansion $\{s_1, \dots, s_P\}$ of $\{z_1, \dots, z_N\}$ and $\{x_1, \dots, x_M\}$ so that

$$\theta = \begin{pmatrix} y'_1 & \dots & y'_P \\ s_1 & \dots & s_P \end{pmatrix} \text{ and } \varphi = \begin{pmatrix} s_1 & \dots & s_P \\ t'_1 & \dots & t'_P \end{pmatrix}$$

and thus

$$\theta\varphi = \begin{pmatrix} y'_1 & \dots & y'_P \\ t'_1 & \dots & t'_P \end{pmatrix}.$$

A relevant subgroup of $G_{n,r}$ is the commutator subgroup, usually denoted by $G_{n,r}^+$. At it was shown in [5] (c.f. [9, Lemma 2.1]), the index of $G_{n,r}^+$ in $G_{n,r}$ is $\gcd(n-1, 2)$, so that $G_{n,r}^+$ coincides with $G_{n,r}$ whenever n is even. For the sake of uniform notation (c.f. [5, Section 5]), we write $G_{n,r}^+ = G_{n,r}$ when n is even.

For any $n \geq 2, r \geq 1$, some interesting features enjoyed by these groups are the following:

- (1) The group $G_{n,r}$ is finitely presented [5, Theorem 4.6].
- (2) The group $G_{n,r}^+$ is simple [5, Theorem 5.4].
- (3) The group $G_{n,r}^+$ contains an isomorphic copy of every countable locally finite group [5, Theorem 6.6].

- (4) The defining relations of the group $G_{n,r}$ are recursively enumerable, so that $G_{n,r}$ has soluble word problem, and thus conjugacy and order soluble problems [5, Section 9].

2. ISOMORPHISMS OF LEAVITT ALGEBRAS

We begin by defining the Leavitt algebras $L_K(1, n)$, which was investigated originally by Leavitt in his seminal paper [6]. For any positive integer $n \geq 2$, and field K , we denote $L_K(1, n)$ by $L_{K,n}$, and call it the Leavitt algebra of type $(1, n-1)$ with coefficients in K . (When K is understood, we denote this algebra simply by L_n). Precisely, $L_{K,n}$ is the quotient of the free associative K -algebra in $2n$ variables:

$$L_{K,n} = K \langle X_1, \dots, X_n, Y_1, \dots, Y_n \rangle / T,$$

where T is the ideal generated by the relations $X_i Y_j - \delta_{ij} 1_K$ (for $1 \leq i, j \leq n$) and $\sum_{j=1}^n Y_j X_j - 1_K$. The images of X_i, Y_i in $L_{K,n}$ are denoted respectively by x_i, y_i . In particular, we have the equalities $x_i y_j = \delta_{ij} 1_K$ and $\sum_{j=1}^n y_j x_j = 1_K$ in L_n . A multiindex will be a sequence $I = \{i_1, \dots, i_k\}$ with $i_j \in \{1, 2, \dots, n\}$ for all $1 \leq j \leq k$. We will then denote $y_I = y_{i_1} y_{i_2} \cdots y_{i_k}$ and $x_I = x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$.

We now fix a fundamental property of L_n that is basic for our purposes.

Lemma 2.1. ([6, Theorem 8]) *Let K be any field, let $n \geq 2$ be a natural number. Then, L_n has module type $(1, n-1)$. In particular, if $r \equiv s \pmod{n-1}$ then $L_n^r \cong L_n^s$ as free left L_n -modules. Consequently, if $r \equiv s \pmod{n-1}$, then there is an isomorphism of matrix rings $M_r(L_n) \cong M_s(L_n)$.*

Remark 2.2. Suppose that $s = r + (n-1)$, and denote $\hat{x} = (x_1, \dots, x_n)$ and $\hat{y} = (y_1, \dots, y_n)$. Then, abovementioned isomorphism is given by the rule

$$\begin{aligned} \varphi : M_r(L_n) &\rightarrow M_s(L_n) \\ A &\mapsto \text{diag}(I_{r-1}, \hat{x}^t) \cdot A \cdot \text{diag}(I_{r-1}, \hat{y}) \end{aligned}$$

In particular, whenever the entries of A has the form $\sum_{i=1}^k y_{I_i} x_{J_i}$ (for $\{I_i, J_i\}_{1 \leq i \leq k}$ sets of multiindices), then so are the entries of $\varphi(A)$. By recurrence on this argument, the same consequence holds for any pair of natural numbers r, s such that $r \equiv s \pmod{n-1}$.

Definition 2.3. For any field K , the extension of the assignments $x_i \mapsto y_i = x_i^*$ and $y_i \mapsto x_i = y_i^*$ for $1 \leq i \leq n$ yields an involution $*$ on $L_K(1, n)$. This involution on $L_K(1, n)$ produces an involution on any sized matrix ring $M_m(L_K(1, n))$ over $L_K(1, n)$ by setting $X^* = (x_{j,i}^*)$ for each $X = (x_{i,j}) \in M_m(L_K(1, n))$. We note that if K is a field with involution (which we also denote by $*$), then a second involution on $L_K(1, n)$ may be defined by extending the assignments $k \mapsto k^*$ for all $k \in K$, $x_i \mapsto y_i = x_i^*$ and $y_i \mapsto x_i = y_i^*$ for $1 \leq i \leq n$. We will say that $X \in M_d(L_n)$ is a unitary provided that $XX^* = X^*X = I_d$, and we will denote by $U(M_d(L_n))$ the group of unitaries of $M_d(L_n)$.

Now, we will quote the essential result of this section

Theorem 2.4. ([1, Theorem 4.14]) *Let d, n be positive integers, and K any field. Let $L_{K,n} = L_n$ denote the Leavitt algebra of type $(1, n-1)$ with coefficients in K . Then $L_n \cong M_d(L_n)$ if and only if $\gcd(d, n-1) = 1$.*

Let us fix the details needed to prove Theorem 2.4, in order to explain why it is so important in the proof of our main result. Essentially, we need to construct a K -algebra isomorphism

$$\begin{aligned} \varphi: L_n &\rightarrow M_d(L_n) \\ x_i &\mapsto X_i \\ y_j &\mapsto Y_j \end{aligned}.$$

Since L_n is a simple algebra, it is enough to fix a set $\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \subset M_d(L_n)$ satisfying the definitory relations of the generators of L_n , and generating $M_d(L_n)$. Now, we present the appropriate $2n$ matrices. For any unital ring R and $i \in \{1, 2, \dots, d\}$ we denote the idempotent $e_{i,i}$ of the matrix ring $M_d(R)$ simply by e_i . We write $n = qd + r$ with $2 \leq r \leq d$. We assume $d < n$, so that $q \geq 1$. The matrices X_1, X_2, \dots, X_q are given as follows. For $1 \leq i \leq q$ we define

$$X_i = \begin{pmatrix} x_{(i-1)d+1} & 0 & 0 \\ x_{(i-1)d+2} & 0 & 0 \\ \vdots & 0 & \dots & 0 \\ x_{id} & 0 & 0 \end{pmatrix} = \sum_{j=1}^d x_{(i-1)d+j} e_{j,1}$$

The two matrices X_{q+1} and X_{q+2} play a pivotal role here. They are defined as follows.

$$\begin{aligned} X_{q+1} &= \begin{pmatrix} x_{qd+1} & 0 & 0 & 0 & 0 & 0 \\ x_{qd+2} & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 \\ x_n & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix} \\ &= \sum_{i=1}^{d-r} e_{i+r,i+1} + \sum_{t=1}^r x_{qd+t} e_{t,1} \end{aligned}$$

and

$$\begin{aligned} X_{q+2} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & a_{q+2,r-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{q+2,r} \\ & & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{q+2,d} \end{pmatrix} \\ &= \sum_{j=1}^{r-2} e_{j,j+s} + \sum_{t=1}^{d-(r-2)} a_{q+2,(r-2)+t} e_{(r-2)+t,d} \end{aligned}$$

(where the elements $a_{q+2,r-1}, a_{q+2,r}, \dots, a_{q+2,d} \in L_n$ are monomials in x -variables). In case $d - r = 0$ or $r - 2 = 0$ we interpret the appropriate sums as zero.

The remaining matrices X_{q+3}, \dots, X_n will have the same general form. In particular, for $q+3 \leq i \leq n$,

$$X_i = \begin{pmatrix} 0 & 0 & a_{i,1} \\ 0 & 0 & a_{i,2} \\ 0 & \dots & \vdots \\ 0 & 0 & a_{i,d} \end{pmatrix} = \sum_{j=1}^d a_{i,j} e_{j,d}$$

(where the elements $a_{i,1}, a_{i,2}, \dots, a_{i,d} \in L_n$ are monomials in the x -variables). In case $q+3 > n$ then we understand that there are no matrices of this latter form in our set of $2n$ matrices. We note that we always have the matrices X_{q+1} and X_{q+2} , since $n = qd + r \geq q \cdot 1 + 2$. We define the matrices Y_i for $1 \leq i \leq n$ by setting $Y_i = X_i^*$.

Now consider this set, which we will call “The List”:

$$\begin{aligned} & x_1^{d-1} \\ & x_2 x_1^{d-2}, x_3 x_1^{d-2}, \dots, x_n x_1^{d-2} \\ & x_2 x_1^{d-3}, x_3 x_1^{d-3}, \dots, x_n x_1^{d-3} \\ & \vdots \\ & x_2 x_1, x_3 x_1, \dots, x_n x_1 \\ & x_2, x_3, \dots, x_n \end{aligned}$$

The key of the proof of Theorem 2.4 is that, whenever $\gcd(d, n-1) = 1$, there is a rule to assign an element in The List to each $a_{i,j}$ in the above set of matrices, in such a way that the resulting set $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ satisfies the definitory relations of the generators of L_n , and generates $M_d(L_n)$. Thus, under such a choice, the above defined map φ is a K -algebra isomorphism.

Remark 2.5. Because of the definition of the above mentioned isomorphism, it is clear that whenever $a \in L_n$ has the form $\sum_{i=1}^k y_{I_i} x_{J_i}$ (for $\{I_i, J_i\}_{1 \leq i \leq k}$ sets of multiindices), then the entries of $\varphi(a)$ have the same form.

We will prove an easy consequence of Theorem 2.4 that will be useful in the sequel. For, we quote the following fact

Lemma 2.6. ([2, Lemma 1]) *Let G be a finitely generated abelian group (written additively). Let $x \in G$ be an element of finite order n , and let $c, d \in \mathbb{N}$. There exists an automorphism $\varphi : G \rightarrow G$ with $\varphi(cx) = dx$ if and only if $\gcd(c, n) = \gcd(d, n)$.*

Corollary 2.7. *Let n, r, s be positive integers, and K any field. Let $L_{K,n} = L_n$ denote the Leavitt algebra of type $(1, n-1)$ with coefficients in K . If $\gcd(r, n-1) = \gcd(s, n-1)$, then $M_r(L_n) \cong M_s(L_n)$.*

Proof. By Lemma 2.6, applied to $G = \mathbb{Z}/(n-1)\mathbb{Z}$, $x = [1] \in \mathbb{Z}/(n-1)\mathbb{Z}$, $c = r$ and $d = s$, there exists a group automorphism $\varphi : \mathbb{Z}/(n-1)\mathbb{Z} \rightarrow \mathbb{Z}/(n-1)\mathbb{Z}$ such that $\varphi([r]) = [s]$. Thus, there exists $l \in \mathbb{N}$ with $\gcd(l, n-1) = 1$ such that $[lr] = [s]$. Since $lr \equiv s \pmod{n-1}$, we have $M_s(L_n) \cong M_r(M_l(L_n))$ by Lemma 2.1. Now, $L_n \cong M_l(L_n)$ by Theorem 2.4, so that $M_r(M_l(L_n)) \cong M_r(L_n)$, which completes the proof. \square

Remark 2.8. Because of Remarks 2.2 and 2.5, the isomorphism given by Corollary 2.7 has the property that, whenever the entries of A has the form $\sum_{i=1}^k y_{I_i} x_{J_i}$ (for $\{I_i, J_i\}_{1 \leq i \leq k}$ sets of multiindices), then so are the entries of $\varphi(A)$. This fact play a role in the proof of the main result of this paper.

3. THE MAIN RESULT

In this section, we will prove the main result of the paper.

Definition 3.1. Let $n \geq 2, r \geq 1$ be natural numbers. We denote by $\mathcal{P}_{n,r}$ the subset of the group $U(M_r(L_n))$ of unitaries of $M_r(L_n)$ composed by matrices in which all the entries are either 0 or have the form

$$\sum_{i=1}^m y_{I_i} x_{J_i},$$

where the I_i, J_i are multiindices.

Lemma 3.2. *For any $n \geq 2, r \geq 1$ natural numbers, $\mathcal{P}_{n,r}$ is a subgroup of $U(M_r(L_n))$.*

Proof. Since $1 = \sum_{i=1}^n y_i x_i$, it is clear that the identity matrix I_r belongs to $\mathcal{P}_{n,r}$, whence it is a nonempty set.

Fix $X, Y \in \mathcal{P}_{n,r}$ two elements. Since Y is an unitary, Y^{-1} is the conjugated transpose of Y (so it lies in $\mathcal{P}_{n,r}$ too), and hence the entries in XY^{-1} are of the form $\sum_{k=1}^r a_{i,k} b_{k,j}$, where

$$a_{i,k} b_{k,j} = \left(\sum_{i=1}^m y_{I_i} x_{J_i} \right) \cdot \left(\sum_{i=1}^{m'} y_{I'_i} x_{J'_i} \right).$$

As

$$(*) \quad x_{J_r} y_{I'_s} = \begin{cases} y_{\widehat{I}_s} & \text{if } I'_s = J_r \widehat{I}_s \\ x_{\widehat{J}_r} & \text{if } J_r = \widehat{J}_r I'_s \\ 0 & \text{otherwise} \end{cases}$$

we conclude that $XY^{-1} \in \mathcal{P}_{n,r}$, as desired. \square

Notice that, if we fix the alphabet $\mathcal{A}_n = \{1, \dots, n\}$, then each multiindex is an element of W_n , and the identity $(*)$ in Lemma 3.2 says that $x_{J_r} y_{I'_s} = 0$ if and only if $x_1 I'_s$ and $x_1 J_r$ are independent elements of $X_1 W_n$. The key for connecting the isomorphism problem of Higman-Thompson groups with Leavitt algebras lies precisely in this fact, that we will exploit.

Now, we will prove a technical results that will be needed later.

Lemma 3.3. *Let $n \geq 2$ be a natural number. If $\alpha = \sum_{i=1}^m y_{I_i} x_{J_i} \in \mathcal{P}_{n,1}$, then both $\{I_1, \dots, I_m\}$ and $\{J_1, \dots, J_m\}$ are expansions of the basis $\{x_1\}$ of $X_1 W_n$ (and thus basis).*

Proof. As the argument is symmetric, we will proof it only for $\{I_1, \dots, I_m\}$.

First, suppose that $\{I_1, \dots, I_m\}$ do not contain a complete expansion of $\{x_1\}$. Then, two different cases could happen:

- (1) The set $\{I_1, \dots, I_m\}$ is independent, and thus can be completed to a basis

$$\{I_1, \dots, I_l, \widehat{I}_{l+1}, \dots, \widehat{I}_r\}.$$

- (2) The set $\{I_1, \dots, I_m\}$ is not independent. So, we can chose a maximal independent subset $\{I_1, \dots, I_l\} \subsetneq \{I_1, \dots, I_m\}$.

Hence, in any of both cases, for $l \leq m$ there exist a maximal independent subset $\{I_1, \dots, I_l\} \subseteq \{I_1, \dots, I_m\}$ and a multiindex Z such that $\{I_1, \dots, I_l, Z\}$ can be expanded to a basis. But then, as Z is independent of the I_j s,

$$0 \neq x_Z = x_Z \cdot 1 = x_Z(\alpha\alpha^*) = (x_Z\alpha)\alpha^* = 0\alpha^* = 0$$

which is impossible.

Now suppose that $\{I_1, \dots, I_m\}$ contains an expansion (i.e. a basis) but it is not a basis, i.e it is not an independent set. Fix $\{I_1, \dots, I_l\} \subsetneq \{I_1, \dots, I_m\}$ a basis, and notice that

$$1 = \left(\sum_{i=1}^l y_{I_i} x_{J_i} \right) \cdot \left(\sum_{j=1}^l y_{J_j} x_{I_j} \right).$$

Hence,

$$\begin{aligned} 1 = \alpha\alpha^* &= \left(\sum_{i=1}^m y_{I_i} x_{J_i} \right) \cdot \left(\sum_{j=1}^m y_{J_j} x_{I_j} \right) \\ &= \left(\sum_{i=1}^l y_{I_i} x_{J_i} + \sum_{i=l+1}^m y_{I_i} x_{J_i} \right) \cdot \left(\sum_{j=1}^l y_{J_j} x_{I_j} + \sum_{j=l+1}^m y_{J_j} x_{I_j} \right) \\ &= 1 + \sum_{i=1}^l \sum_{j=l+1}^m y_{I_i} x_{J_i} y_{J_j} x_{I_j} + \sum_{i=l+1}^m \sum_{j=1}^l y_{I_i} x_{J_i} y_{J_j} x_{I_j} + \sum_{i=l+1}^m \sum_{j=l+1}^m y_{I_i} x_{J_i} y_{J_j} x_{I_j}. \end{aligned}$$

Thus, the last 3 summands equal zero, and in particular for any $l+1 \leq i \leq m$ we have $0 = y_{I_i} x_{J_i} y_{J_i} x_{I_i} = y_{I_i} x_{I_i}$, which is impossible. So, we are done. \square

The goal is to prove that for any $n \geq 2, r \geq 1$, $\mathcal{P}_{n,r} \cong G_{n,r}$. In order to do more comprehensible the argument, first we will prove the result in the particular case $r = 1$. This result is analogous to [8, Proposition 9.6], but the proof is different.

Proposition 3.4. *If $n \geq 2$ is a natural number, then $\mathcal{P}_{n,1} \cong G_{n,1}$.*

Proof. By [5, Lemma 4.1], given an element $x \in G_{n,1}$, we can express it by using a symbol

$$x = \begin{pmatrix} I_1 & \dots & I_m \\ J_1 & \dots & J_m \end{pmatrix}$$

where both $\{I_1, \dots, I_m\}$ and $\{J_1, \dots, J_m\}$ are expansions of the basis $\{x_1\}$ of $X_1 W_n$ (and thus basis). Now define

$$\alpha_x = \sum_{i=1}^m y_{I_i} x_{J_i} \in L_n.$$

Notice that

$$\alpha_x \alpha_x^* = \left(\sum_{i=1}^m y_{I_i} x_{J_i} \right) \cdot \left(\sum_{j=1}^m y_{J_j} x_{I_j} \right) = \sum_{i=1}^m y_{I_i} x_{I_i} = 1,$$

where the last two equalities are due to the fact that both $\{I_1, \dots, I_m\}$ and $\{J_1, \dots, J_m\}$ are expansions of the basis $\{x_1\}$ of $W_{n,1}$. Similarly, $\alpha_x^* \alpha_x = 1$, so that $\alpha_x \in \mathcal{P}_{n,1}$. Define a map

$$\begin{aligned} \varphi : G_{n,1} &\rightarrow \mathcal{P}_{n,1} \\ x &\mapsto \alpha_x \end{aligned},$$

and notice that φ send symbols equivalent by elementary expansions to the same element in L_n . Thus, if

$$x = \begin{pmatrix} I_1 & \dots & I_m \\ J_1 & \dots & J_m \end{pmatrix} \text{ and } y = \begin{pmatrix} R_1 & \dots & R_k \\ S_1 & \dots & S_k \end{pmatrix},$$

again by [5, Corollary 1 to Lemma 2.4] there exists a common expansion $\{J'_1, \dots, J'_t\}$ of both $\{J_1, \dots, J_m\}$ and $\{R_1, \dots, R_k\}$ such that

$$x = \begin{pmatrix} I'_1 & \dots & I'_t \\ J'_1 & \dots & J'_t \end{pmatrix} \text{ and } y = \begin{pmatrix} J'_1 & \dots & J'_t \\ S'_1 & \dots & S'_t \end{pmatrix},$$

whence

$$xy = \begin{pmatrix} I'_1 & \dots & I'_t \\ S'_1 & \dots & S'_t \end{pmatrix}.$$

By the above remark, we get $\varphi(xy) = \varphi(x)\varphi(y)$, so that φ is a group morphism.

Now, if $\alpha = \sum_{i=1}^m y_{I_i} x_{J_i} \in \mathcal{P}_{n,1}$, then the element

$$x_\alpha = \begin{pmatrix} I_1 & \dots & I_m \\ J_1 & \dots & J_m \end{pmatrix}$$

belong to $G_{n,1}$ by Lemma 3.3, so that

$$\begin{aligned} \psi : \mathcal{P}_{n,1} &\rightarrow G_{n,1} \\ \alpha &\mapsto x_\alpha \end{aligned}$$

is a well-defined map. Moreover, $\varphi(x_\alpha) = \alpha$, so that φ is an onto map. As ψ is clearly compatible with the equivalence of symbols by elementary expansions, it turns out that ψ is a group morphism. A simple inspection shows that φ and ψ are mutually inverses, so we are done. \square

Now, we prove the general version.

Proposition 3.5. *If $n \geq 2$ and $r \geq 1$ are natural numbers, then $\mathcal{P}_{n,r} \cong G_{n,r}$.*

Proof. Consider $X_r = \{x_1, \dots, x_r\}$ as basis of $X_r W_n$, and take

$$x = \begin{pmatrix} I_1 & \dots & I_k \\ J_1 & \dots & J_k \end{pmatrix} \in G_{n,r}.$$

For any $1 \leq i, j \leq r$, consider the strictly ascending finite sequence

$$1 \leq l(i, j)_1 < \dots < l(i, j)_{s(i, j)} \leq k$$

such that $I_{l(i,j)_t}$ starts in x_i and $J_{l(i,j)_t}$ starts in x_j for every $1 \leq t \leq s(i,j)$ (Notice that it can happens for some sequences in this list to be empty). Consider now the multiindices $I'_{l(i,j)_t}$ and $J'_{l(i,j)_t}$ obtained from $I_{l(i,j)_t}$ and $J_{l(i,j)_t}$ (respectively) by erasing the initial x_i (resp. x_j). Now we define the matrix $X \in M_r(L_n)$ whose (i,j) -entry is

$$X_{i,j} = \sum_{p=1}^{s(i,j)} y_{I'_{l(i,j)_p}} x_{J'_{l(i,j)_p}}.$$

Notice that $x_{J'_{l(i,k)_p}} y_{J'_{l(j,k)_q}} = \delta_{i,j} \cdot \delta_{p,q}$; indeed, if $x_{J'_{l(i,k)_p}} y_{J'_{l(j,k)_q}} = 1$ for $i \neq j$ or $p \neq q$, then the symbol of x shall contain entries

$$x = \begin{pmatrix} \cdots & x_i I'_{l(i,k)_p} & \cdots & x_j I'_{l(j,k)_q} & \cdots \\ \cdots & x_k J'_{l(i,k)_p} & \cdots & x_k J'_{l(j,k)_q} & \cdots \end{pmatrix}$$

with $x_k J'_{l(i,k)_p} = x_k J'_{l(j,k)_q}$, which is impossible by definition of symbol. Also, since $\{I_1, \dots, I_k\}$ is an expansion of X_r ,

$$\{I_{l(i,1)_1}, \dots, I_{l(i,1)_{s(i,1)}}, \dots, I_{l(i,r)_1}, \dots, I_{l(i,r)_{s(i,r)}}\}$$

is an expansion of the element $x_i \in X_r$.

We then have

$$\begin{aligned} (XX^*)_{i,j} &= \sum_{k=1}^r \left(\sum_{p=1}^{s(i,k)} y_{I'_{l(i,k)_p}} x_{J'_{l(i,k)_p}} \right) \cdot \left(\sum_{q=1}^{s(j,k)} y_{J'_{l(j,k)_q}} x_{I'_{l(j,k)_q}} \right) \\ &= \delta_{i,j} \cdot \sum_{k=1}^r \sum_{p=1}^{s(i,k)} y_{I'_{l(i,k)_p}} x_{I'_{l(i,k)_p}} = \delta_{i,j}, \end{aligned}$$

and similarly $(X^*X)_{i,j} = \delta_{i,j}$. Hence, $X \in \mathcal{P}_{n,r}$. Thus,

$$\begin{array}{ccc} \varphi : & G_{n,r} & \rightarrow \mathcal{P}_{n,r} \\ & x & \mapsto X \end{array}$$

is a well-defined map. Clearly φ respects the equivalence of symbols by elementary expansions in $G_{n,r}$, so that it is straightforward but tedious to prove that in fact it is a group morphism.

Now, take $X \in \mathcal{P}_{n,r}$. For any $1 \leq i, j \leq r$,

$$X_{i,j} = \sum_{p=1}^{s(i,j)} y_{I_{l(i,j)_p}} x_{J_{l(i,j)_p}}$$

for suitable sets of multiindices. We will show that both $W_I = \left\{ \{x_i I_{l(i,j)_p}\}_{1 \leq p \leq s(i,j)} \right\}_{1 \leq i, j \leq r}$ and $W_J = \left\{ \{x_j J_{l(i,j)_p}\}_{1 \leq p \leq s(i,j)} \right\}_{1 \leq i, j \leq r}$ are expansions of the basis X_r . Notice that for any $1 \leq i \leq r$,

$$(*) \quad 1 = \sum_{k=1}^r X_{i,k} X_{i,k}^* = \sum_{k=1}^r \left(\sum_{p,q=1}^{s(i,k)} y_{I_{l(i,k)_p}} x_{J_{l(i,k)_p}} y_{J_{l(i,k)_q}} x_{I_{l(i,k)_q}} \right).$$

Fix any $1 \leq i \leq r$, and let $W_I(i) = \left\{ \left\{ x_i I_{l(i,j)_p} \right\}_{1 \leq p \leq s(i,j)} \right\}_{1 \leq j \leq r}$. If it do not contain a complete expansion of x_i , then the same argument as in Lemma 3.3 shows that there exist a maximal independent subset W' of $W_I(i)$ and a multiindex Z such that $W' \cup \{Z\}$ is a part of a basis for x_i . Hence,

$$0 \neq x_Z \cdot e_{i,i} = (x_Z \cdot e_{i,i})(XX^*) = ((x_Z \cdot e_{i,i}X)X^*)$$

has (k, j) -entry equal to zero for any $k \neq i$ and any j , while

$$(x_Z \cdot e_{i,i}X)_{i,k} = \sum_{p=1}^{s(i,k)} x_Z y_{I_{l(i,k)_p}} x_{J_{l(i,k)_p}} = 0,$$

which is impossible. On the other side, if $W_I(i)$ contains an expansion of $\{x_i\}$ but it is not a basis, again the argument in Lemma 3.3 and the identity $(*)$ give us a contradiction. Thus, $W_I(i)$ is an expansion of x_i , and thus so is W_I of X_r . Similarly we get that W_J is an expansion of X_r . Since both sets has the same cardinality,

$$x_X = \begin{pmatrix} x_1 I_{l(1,1)_1} & \cdots & x_i I_{l(i,j)_p} & \cdots & x_r I_{l(r,r)_{s(r,r)}} \\ x_1 J_{l(1,1)_1} & \cdots & x_j J_{l(i,j)_p} & \cdots & x_r J_{l(r,r)_{s(r,r)}} \end{pmatrix}$$

is a symbol of an element of $G_{n,r}$, so that

$$\begin{aligned} \psi : \mathcal{P}_{n,r} &\rightarrow G_{n,r} \\ X &\mapsto x_X \end{aligned}$$

is a well-defined map. Moreover, $\varphi(x_X) = X$, so that φ is an onto map. As ψ clearly respects the equivalence of symbols by elementary expansions, ψ is a group morphism, and φ and ψ are mutually inverses, so we are done. \square

Now, we are ready to prove the main result in the paper.

Theorem 3.6. *Let $n, m \geq 2$ and $r, s \geq 1$ be natural numbers. Then, $G_{m,r}^+ \cong G_{n,s}^+$ if and only if $m = n$ and $\gcd(n-1, r) = \gcd(n-1, s)$.*

Proof. The “only if” part is [5, Theorem 6.4].

Now, assume that $\gcd(n-1, r) = \gcd(n-1, s)$. Then, by Corollary 2.7, there exists a K -algebra isomorphism $\varphi : M_r(L_n) \rightarrow M_s(L_n)$ that, by Remark 2.8, restricts to a group isomorphism $\phi : \mathcal{P}_{n,r} \rightarrow \mathcal{P}_{n,s}$. As $G_{n,r} \cong \mathcal{P}_{n,r}$ and $G_{n,s} \cong \mathcal{P}_{n,s}$ by Proposition 3.5, we conclude that $G_{n,r} \cong G_{n,s}$, and thus $G_{n,r}^+ \cong G_{n,s}^+$, as desired. \square

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